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Research articles

Characteristics of pre-service mathematics teacher when solving convergent sequence problems

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Abstrak Barisan konvergen merupakan salah satu konsep yang sulit dipahami peserta didik pada matakuliah analisis real. Pembelajaran berbasis pemecahan masalah dapat menjadi alternatif untuk menanamkan konsep barisan konvergen. Penelitian ini bertujuan untuk mendeskripsikan aktivitas peserta didik dalam memecahkan masalah barisan konvergen berdasarkan representasi matematis yang mereka kembangkan. Kajian ini menggunakan pendekatan kualitatif dengan jenis studi kasus yang melibatkan 14 responden yang dipilih secara *purposive sampling*. Data dikumpulkan menggunakan instrumen tes dan pedoman wawancara. Hasil penelitian ini menunjukkan bahwa responden dapat dikategorikan menjadi kelompok representasi dominan dan kelompok representasi tidak dominan. Responden yang dominan verbal cenderung menyampaikan ide melalui kata-kata yang jelas, mampu merincikan konsep serta menyusunnya dengan argumen dan pemikiran yang logis. Responden yang dominan visual cenderung menginterpretasikan atau menerjemahkan tampilan visual dalam membangun tahapan pemecahan masalah. Adapun responden dominan simbolik cenderung memecahkan masalah dengan menguraikannya menjadi beberapa langkah penyelesaian yang terstruktur secara konseptual. Karakteristik responden yang tidak dominan pada satu tipe representasi menunjukkan fleksibilitas mahasiswa dalam memahami masalah dengan menggunakan beragam representasi yang disesuaikan dengan kondisi dan pengetahuan yang dimiliki. Temuan penelitian ini dapat menjadi acuan untuk merancang skenario pembelajaran berdasarkan karakteristik individu dalam pemecahan masalah barisan konvergen.

Kata kunci *Pendekatan pemecahan masalah, Representasi matematis, Barisan konvergen, Studi kasus*

Abstract Convergent sequences pose a challenge for students to comprehend in real analysis courses. Problem-solving based learning can serve as an alternative approach for imparting the understanding of convergent sequences. The objective of this research is to provide a description of the activities undertaken by students when solving problems related to convergent sequences, focusing on specific characteristics of their problem-solving approaches. This study used a qualitative methodology, namely a case study design, with a sample size of 14 participants who were recruited using purposive sampling. The collection of data was conducted via tests and interviews. The findings of this study indicate that the participants can be classified into dominant representation groups and non-dominant representation groups. The subjects who are verbally dominant tend to express ideas using precise language, possess the ability to elaborate on concepts, and demonstrate logical reasoning and argumentation skills. The subjects who are visually dominant tend to analyze or convert visual representations throughout the process of problem-solving. Meanwhile, those who are symbolically dominant tend to approach problem-solving by breaking down the difficulties into multiple solution phases that are conceptually organized. The subjects who are not dominant in a certain type of representation show flexibility in understanding the problem by using a variety of representations that are appropriate for their situation and level of knowledge. The results of this study can serve as

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a guide for constructing educational approaches, taking into account the characteristics of students when solving problems related to convergent sequences.

Keywords *Problem solving approach, Mathematical representation, Convergent sequence, Case study*

Introduction

Higher education mathematics has complex characteristics and involves a high level of abstraction (Çelik & Özdemir, 2020; Maslihah et al., 2020), with more sophisticated and abstract materials compared to those of secondary school mathematics (Langoban, 2020; Lima et al., 2019). At the tertiary level, students not only learn basic concepts but also acquire knowledge and thinking skills to analyze and solve problems, as well as develop valid arguments (Adeoye & Jimoh, 2023; Angraini & Wahyuni, 2021; Iwuanyanwu, 2022; Kola & Molise, 2023). One of the suitable courses for these purposes is real analysis.

In real analysis courses, the limit of a sequence describes the behavior of the sequence when its index approaches infinity. By paying attention to how the sequence elements approach a certain value, it can be concluded whether it has a limit or converges (Arnal-Palacián et al., 2020; Tuna et al., 2019).

Teaching abstract concepts such as convergent sequences poses its own challenges, including students' limited understanding of how to prove mathematical theorems and propositions (Sebsibe & Feza, 2020; Subedi, 2020; Widiati & Sthephani, 2018), leading them to answer unclearly or incorrectly (Isnani et al., 2021; Kristianto et al., 2019). In producing proofs, it is necessary to understand the concepts involved by using mathematical arguments and proof procedures that connect mathematical ideas correctly (Feriyanto, 2017; Herizal et al., 2019). This guides students to develop the ability to manipulate limit concepts correctly by carefully applying limit properties and operations, limit theorems, and the epsilon-delta principle (Nurdin et al., 2021).

Recognizing student abilities is crucial for the teaching of abstract concepts such as convergent sequences. As educators, lecturers need to provide the necessary support for interesting interactive learning activities that address students' needs (Munna & Kalam, 2021), e.g., by using mathematical representation. This makes the learning process more effective in helping students comprehend the material. Mathematical representation is the ability to identify various aspects of a problem, present the problem symbolically or in the form of a mathematical model, choose a solution strategy, carry out algorithmic activities, and make interpretations both orally and in writing (Ratumanan et al., 2022). The examination of the characteristics of these representations involves analyzing how students represent mathematical concepts, especially in the context of convergent sequences. This analysis is carried out by identifying representation variables, including symbolic, visual and verbal representations (L.Man et al., 2022).

The results of previous research show that representational abilities contribute to facilitating students' comprehension of concepts and further improving their abilities (Afriyani et al., 2018; Prayitno et al., 2021). The final ability that is expected to be acquired by studying the limits of a sequence is that students can explain sequences of real numbers and related theorems while becoming adept when applying them to solve problems. Therefore, it is necessary to take a holistic approach to integrating mathematical concepts as a whole so that students can see the relationship between ideas in a broader mathematical context (Rosa et al., 2016). Holistic mathematical representation can be an integral part of the approach to studying the limits of sequences, especially convergent sequences. This study examines the use of mathematical representations to define concepts in a more comprehensive and inclusive way so as to provide solutions to improve mathematical understanding (Mainali, 2021), allowing students to intuitively see and understand the behavior of a sequence as it approaches a limit.

Through various mathematical representations, this study supports Gardner's theory of *multiple intelligences* which emphasizes the importance of presenting information in diverse ways (Diarni et al., 2023; Peňalber, 2023) as well as the constructivism theory which emphasizes students' active role in gaining knowledge or understanding by interacting directly with mathematical representations (Kumar Shah, 2019; Vintere, 2018).

This study not only examines students who are dominant in one type of mathematical representation but also observes those who have more than one type of representation ability. An in-depth analysis was made to obtain further information regarding a learning process that is more inclusive and responsive to the needs of diverse students which has not been explored much in previous research.

Theoretical review

Convergence of a sequence

A sequence is said to be convergent if its limit exists. In real analysis, convergence means how a sequence approaches a certain value as the index increases towards infinity. Understanding and mastering this concept allows students to learn and apply several basic theories such as logic and calculus. The formal definition of the limit of a sequence is as follows, A sequence $X = (a_n)$ in $\mathbb{R}$ is said to be convergent to $a \in \mathbb{R}$, or there is a limit of (a_n) if for every $\varepsilon > 0$, there exists a natural number $K(\varepsilon)$ such that $|a_n - a| < \varepsilon$ for all $n \geq K(\varepsilon)$ (Bartle & Sherbert, 2011; Ponnusamy, 2012; Rabih, 2017).

The above definition confirms that if a is the limit of a sequence (a_n), then (a_n) which converges to a is written as $\lim_{n\to\infty}(a_n) = a$, $\lim(a_n) = a$, or $a_n \to a$. If (a_n) is not convergent, then, it is divergent. In other words, a sequence is said to be convergent if it has a limit; if not, the sequence is said to be divergent.

The example of convergent sequences is as follows: if $a_n = c$ for every n, then $a_n \to c$. In fact, $\varepsilon > 0$ can be given by taking $N = 1$. Therefore, $|a_n - c| = 0$ for all $n \geq N = 1$. Another example is $a_n = \frac{1}{n} \to 0$. Meanwhile, the example of divergent sequences is $a_n = n, a_n = \ln(n), a_n = -n^2$, or $a_n = 2^n$, where the sequence has infinitely larger or smaller values when $n \to \infty$ (Arnal-Palacián et al., 2020).

Convergent sequences have a concept similar to that of the limit of the function $f(x)$ if $x \to \infty$ as in Calculus I or Differential Calculus. To illustrate this, the graph of the sequence $a_n = 1 + \frac{2}{n}$ and the graph of the function $f(x) = 1 + \frac{2}{x}$ are displayed in Figure 1, both of which converge to 1 as n and x approaches ∞. The idea of $\lim_{n\to\infty}\frac{1}{n} = 0$ can also be used to solve these problems.

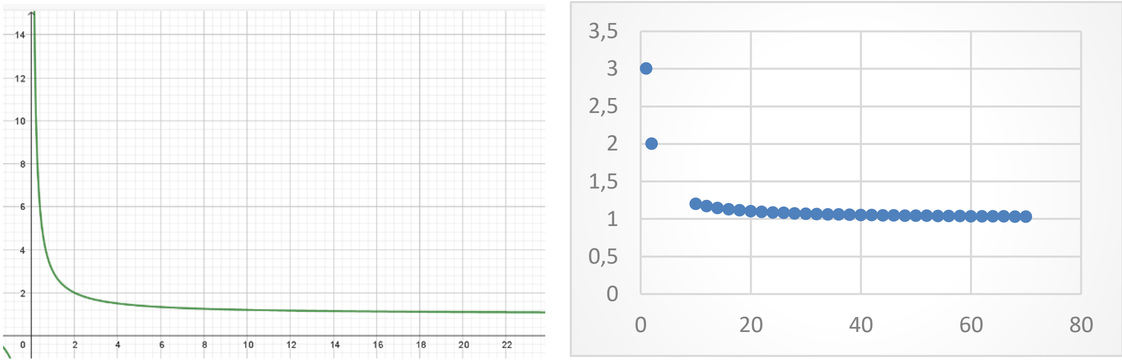

Figure 1. Graph of $f(x) = 1 + \frac{2}{x}$ (left panel) and $a_n = 1 + \frac{2}{n}$ (right panel)

In proving a sequence that converges to a certain value, the terms of the sequence (a_n) if $n \to \infty$ must be understood first so that the value of the limit a can be guessed. In some cases, the form of (a_n) is not always easy. Thus, when solving the limit of a sequence, it is necessary to remember related concepts such as the use of L'Hospital's rule and the properties of convergent sequences (Ross, 2013).

Students' comprehension of the convergence of a sequence or limit of a function directs them to understand the proof of limits, where the lecturer needs to involve a series of steps that are expected to help students grasp the formal definition of the limit of a sequence, identify sequence properties, take ε and determine N, carry out a preliminary analysis, choose a method of proof, and make conclusions (Schüler-Meyer, 2020; Tuna et al., 2019).

Mathematical representation

The importance of representation in mathematics teaching and learning has been widely recognized, including by the National Council of Teachers of Mathematics (NCTM) which has made representation one of the standard processes in the school mathematics curriculum (Mainali, 2021). Therefore, the use of appropriate and effective representations is key in facilitating deep understanding and mastery of mathematical concepts by students.

The application of representations offered by NCTM is emphasized so that students are able to: (a) create and use representations to organize, record, and communicate ideas; (b) develop embodied mathematical representations that can be applied flexibly and precisely for certain purposes; and (c) communicate representations to model and interpret physical, social, and mathematical phenomena (Novianti & Retnawati, 2019). Characteristics of mathematical representation refer to the way a concept or object is modeled or represented. Representations are generally presented in the form of mathematical ideas as a step in solving problems (Hadiastuti et al., 2019).

Representation ability is students' skill in expressing mathematical ideas using certain methods that are considered to help them solve problems more easily (Ratumanan et al., 2022). In this regard, the forms of students' representation are likely to vary depending on their abilities and preferences in conveying their ideas either visually, verbally, or symbolically. (Hariyani et al., 2023). The indicators for the three representations are presented in Table 1.

Table 1. Indicators of representational abilities (Utomo & Syarifah, 2021)

No	Representation	Indicator
1	Visual	Presenting problems in the form of images to solve them
2	Symbolic	Using mathematical expressions in symbols or numbers to solve problems
3	Verbal	Writing an interpretation of the problem presented

Each representation has a specific purpose that shows the characteristics of the student to minimize ambiguity and ensure that the information can be understood clearly. For those who are unfamiliar with the concept, the representation is presented in a way that makes it easy to understand and interpret (Afriyani et al., 2018; Hadiastuti et al., 2019). The flexibility of a good representation allows it to be used in a variety of contexts, purposes, or applications.

Methods

This study uses a qualitative approach with a case study design (Creswell & Creswell, 2018; Rashid et al., 2019). The procedure for conducting this study is described in Figure 2.

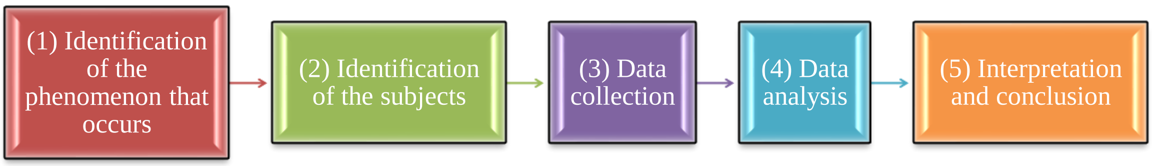

Figure 2. Research procedure (Fraenkel et al., 2012)

The subjects were all students of Mathematics Education Department, Faculty of Education and Teacher Training, Universitas Islam Negeri Datokarama Palu who took the real analysis course and met the purposive sampling criteria, totaling 14 students. The samples were selected with the consideration that the students had studied the basic concepts of limits and sequences and were willing to be fully involved in providing information related to the solutions given.

The data was collected through written tests and interviews. The test given is related to proving the limit of a sequence by considering the assessment of mathematical representation, with visual, symbolic, and verbal representations indicator as shown in Table 1 (Utomo & Syarifah, 2021). The test instruments used (see Figure 3) have passed the logical and empirical validity tests, with a high average validity value of 0.75, a very high level of reliability with a value of r = 0.95, and a moderate difficulty level of both problems A and B with values of 0.48 and 0.59, respectively. Meanwhile, the index of discriminating power of problem A is 0.35, indicating test items with little or no revision, whereas that of problem B is 0.46, indicating test items that function well. The rubric used in assessing mathematical representations is given in Table 2.

How do you prove the following limits?

A) $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}$,

B) $\lim_{n\to\infty}\left(\frac{n^2}{n!}\right)=0$

Figure 3. Written test instrument

Table 2. Scoring rubric for mathematical representation

Assessment criteria	Score	Description
The use of representation in solving problems	1	The answer or information provided does not match the problem
	2	The answer or information provided is incomplete
	3	The answer or information provided is quite complete but not systematic
	4	The answer or information provided is complete and systematic

The interview instrument was semi-structured and used to find out more about the student's explanation when dealing with convergent sequence problems according to their mathematical representation. The instrument was assessed by 3 experts based on three aspects, i.e., construction, language and content validity, which overall obtained score is 2.33 showing the feasibility of the instrument to use with revision.

Findings

The variations in students' characteristics in representing convergent sequences can provide an overview of diverse aspects of students' mathematical understanding of their ability to prove the limits of these sequences. Such variations are visible in the results of the test in which 14 subjects were required to prove the limits of two sequences in 30 minutes. The written test outcomes are summarized in Table 3.

Table 3. Details of representation ability in proving the limit of a sequence

Problem	Verbal representation	Visual representation	Symbolic representation	Unable to solve
A	3 students (HP, MJ, WN)	5 students (NU, IS, NI, ES, NS)	4 students (IN, TR, EF, YA)	2 students (RM, HM)
B	3 students (HP, MJ, YA)	6 students (WN, NU, IS, NI, ES, EF)	2 students (IN, TR)	3 students (RM, HM, NS)

Table 3 shows students' performance in solving or proving the convergent sequence problems in Figure 3. Based on this table, two variations need to be studied in depth, namely subjects who are dominant in each type of representation and those who do not have dominant representation. In this study, students' dominant type of mathematical representation refers to their tendency to use certain methods or forms of representation in understanding and solving mathematical problems. If students solve the problems using more than one type of representation, this condition is regarded as non-dominant.

How do the subjects in the dominant group solve the problem?

Verbal approach

The subject HP (and MJ) used verbal representation to prove the limits of the sequences. HP's test results are displayed in Figure 4.

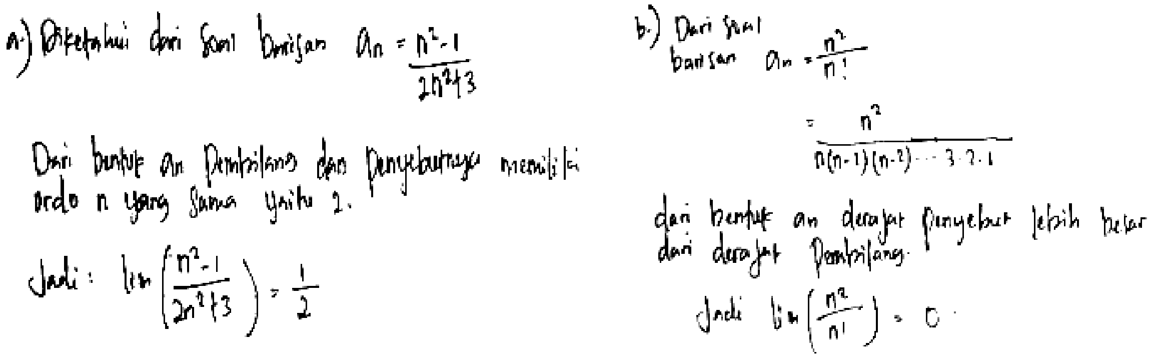

Figure 4. HP's test outcomes for problem A and B

HP's test results have the following implications. *First*, HP presents problems according to the representation she has mastered, which is verbal representation. She explains her understanding of the relationship between keywords in problems and answer ideas. The following is an excerpt from an interview with HP:

> *"In my opinion, Ma'am* […] *conveying ideas this way (verbally) helps me understand [the problems] more easily. Because I follow my thoughts and I try to connect them with my knowledge, Ma'am."*

Second, by understanding the clear process and line of thinking, HP can explain the steps needed to prove the limit of a sequence with clarity using precise and easy-to-understand language so that the narrative conveyed is very comprehensive. This indicates HP's excellent communication skills, as described in the following excerpt:

> *"For me, Ma'am* […] *I have always studied mathematics, which is full of mathematical notations and formulas, using sentences that are easy to understand. So* […] *as the problems you gave… I solved them by focusing on the keyword, that is* a_n*. Problems A and B are both rational, meaning I only* [*need to*] *pay attention to the algebraic form of the numerator and denominator, Ma'am* […] *Then, I identified which one has a higher degree* […] *In problem A, both the numerator and the denominator have the same degree of* n*, so the result is the same, that is,*$1/2$*. As for problem B* […] *the numerator* n^2 *has degree 2, and the denominator is* $n!$ […] *while* $n! = n(n-1)(n-2)(n-3) \ldots 3.2.1.$*, meaning that the degree of* n *is more than 2, Ma'am* […] *So, if it is divided, there are remaining degrees in the denominator* […] *that's until the limit is equal to 0."*

Apart from the strengths exploited by HP in using verbal representations, her knowledge of the model or form of the sequence a_n is explored in this study. HP is hesitant in presenting a sketch of sequence a_n. Even though HP shows the movement of the dots for the sequence in problem A correctly, she is less precise in determining the position of several points in problem B, as seen in Figure 5.

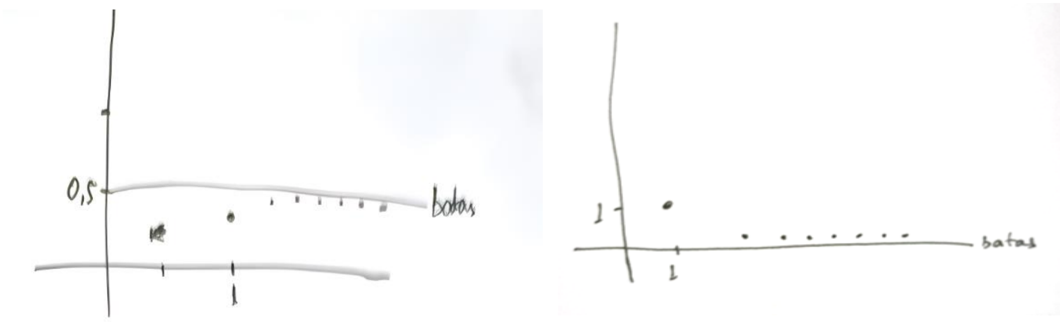

Figure 5. HP's sketches regarding sequences $\left(\frac{n^2-1}{2n^2+3}\right)$ and $\left(\frac{n^2}{n!}\right)$

Thus, subject HP has the ability to convey ideas clearly and effectively, as well as to detail and organize concepts with logical arguments and thinking. However, he is barely capable of writing symbols or algebraic forms of a_n and drawing geometric shapes, signifying his weakness in aspects of abstract concepts.

Visual approach

Along with NU, NI, and ES, the subject IS used visual representation to prove the limits of the sequences. IS's test results are as displayed in Figure 6.

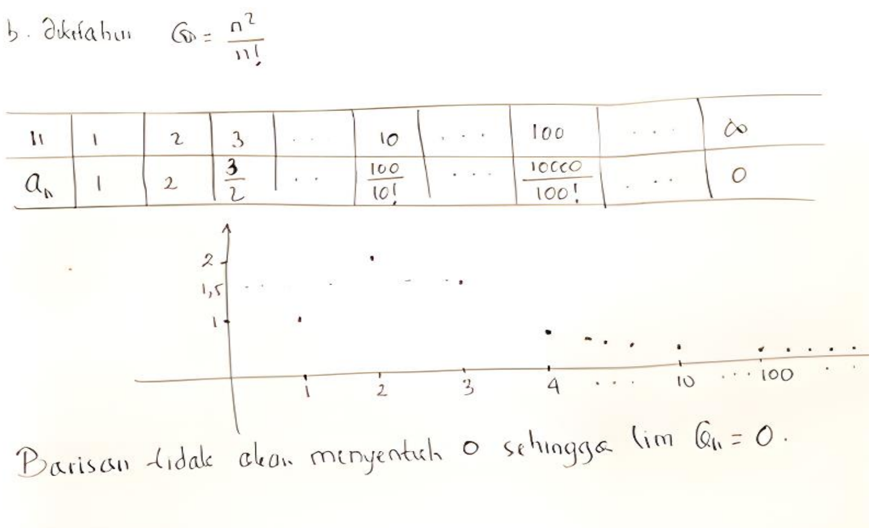

Figure 6. IS's test outcome for problem B

IS's test results have the following implications: *First*, IS has the ability to understand and present mathematical concepts using visual representations. She is skilled at creating graphs that visualize how a sequence approaches or reaches a limit. With good visual intuition, IS understands the pattern of the graphic representation to be created, as shown in the following excerpt:

> *"I saw the shape (while pointing to the writing a_n on the answer sheet), where a_n looks like a function, Ma'am* […] *Thus, my first idea was to draw a graph from the table* […] *I didn't use GeoGebra, Ma'am* […] *so I used a table to draw the graph."*

Based on the interview, IS used a tabulation approach as the first step to present the solution by paying attention to the details of each approach taken to the number n. *Second*, IS is also able to proficiently read and interpret the data contained in tables which can help understand changes and patterns in limit of sequence problems. The following is an excerpt from an interview with IS:

> *"*[*In these problems,*] *n is a natural number whose domain is 1, 2, 3, and so on* […] *So, I substituted it into the formula a_n… thus getting* [*these results*], *Ma'am… (while pointing out that each n gets a single a_n)* […] *and I tried to take easy numbers like 10, 100, and so on* […] *I didn't continue with large numbers; I immediately drew conclusions because the value of a_n is almost the same. So, the conclusions are that* [*the limit of the sequence in*] *problem A is $\frac{1}{2}$ and* [*that of*] *problem B is 0."*

IS tends to show the ability to understand geometric concepts related to the limit of sequence problems. She creatively explores visual representations by understanding how to relate existing information (sequence patterns) with table and image approaches. However, based on IS's writing, it can be seen that the visual representation presented is unelaborate, less formal, and has minimal mathematical notation, which may lead to ambiguity in developing mathematical understanding.

Symbolic approach

TR (and IN) used symbolic representation to prove the limits of the sequences. TR's test results are displayed in Figure 7.

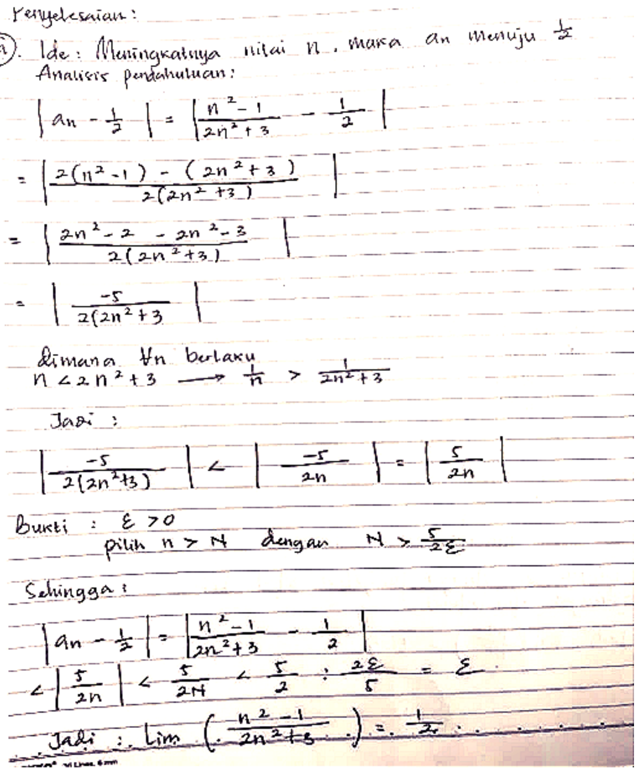

Figure 7. TR's test outcome for problem A

As seen, TR demonstrates the ability to use mathematical notation and symbols effectively in presenting the steps to prove the limit of a sequence. TR understands the definitions and theorems related to the limit of a sequence along with the use of mathematical notation on it. The following is an excerpt from an interview with TR:

> *"I solved the problems by following the definition of the limit of a sequence, Ma'am, which states that* […] *for example, for every positive epsilon, there is N so that if n is greater than N then it is that $|a_n - a| <$ epsilon…Then, for problem A, a preliminary analysis was initially carried out, Ma'am, to find the appropriate N value* […] *thus I solved the problem, Ma'am. That's how I got $N > \frac{5}{2\varepsilon}$. After that* […] *I moved on to the proof, Ma'am* […] *according to the definition, I started by taking positive epsilon and choosing $n > N$ with $N > \frac{5}{2\varepsilon}$ such that the direction aligns with the preliminary analysis, Ma'am…it is proven that $\left|a_n - \frac{1}{2}\right| <$ epsilon, thus showing that $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$… As for problem B, Ma'am… the idea is similar to that of problem A … I got $N > \frac{2}{\varepsilon}$ to later prove that $|a_n - 0| <$ epsilon, Ma'am. Therefore, $\lim_{n\to\infty}\left(\frac{n^2}{n!}\right) = 0$."*

Based on her explanation, TR has a strong understanding of the mathematical concepts involved in the stage of proving the convergence of a sequence, especially the concept of approximation in determining the value of *N*. TR demonstrates the ability to formulate arguments formally (following definitions); she understands how to effectively use notation to prove or explain concepts in solving the limit of a sequence. This results in TR being quite confident in presenting complex ideas correctly and consistently. However, TR seemed to experience problems when asked to present these ideas geometrically.

How do the subjects in the non-dominant group solve the problem?

Verbal and visual-verbal approach

The subject WN solved problem A using a verbal approach and problem B using a visual-verbal approach. WN's test results are displayed in Figure 8.

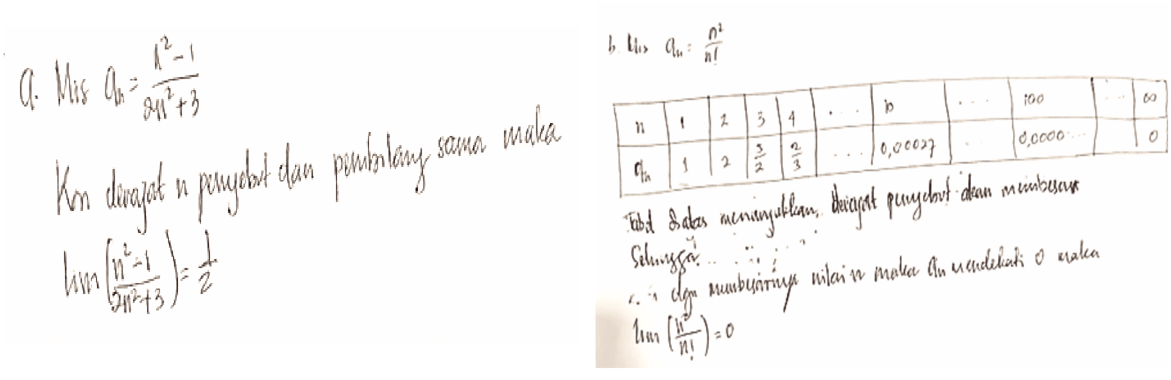

Figure 8. WN's test outcomes for problem A and B

Based on the test results, WN tends to explain what is seen in the table using words. This means that WN integrates verbal and visual representations in conveying her understanding, showing that WN is able to respond to questions well, both verbally and visually (by using tabulation). These results are supported by an excerpt from an interview with WN about the ability to visually-verbally represent the limit of sequence problems as follows:

> *"As for problem A, Ma'am […] I immediately saw the a_n model, Ma'am […] luckily, it's quite easy […] I followed the method in Differential Calculus, Ma'am […] to find an infinite limit and the form is like that, Ma'am (pointing to a_n) […] look at the degree of the denominator where there is n, right, Ma'am […] So I viewed it as the same as the numerator […] so, that's how it is, Ma'am, the result is $\frac{1}{2}$ […] Meanwhile, problem B has a factorial, Ma'am […] I was confused at first […] If I directly explained the n factorial, Ma'am […] the degree of the denominator would not be visible, Ma'am. In problem B, Ma'am […] a_n equals $\frac{n^2}{n!}$ […] so I made a table by providing the details of n = 1, 2, 3, and so on […] Then, I put the n value into the a_n formula, Ma'am […] for $n = 1, a_1 = 1, n = 2, a_2 = 2$, and so on […] such that for a large n, a_n becomes very small towards 0."*

Based on the interview, WN is able to verbally explain the steps for solving problems using mathematical terms clearly and precisely; she even simplifies the steps for solving limit of sequence problems to make them simpler and easier to understand. Apart from the strengths exploited by WN, she also has weaknesses in the aspect of conveying ideas thoroughly; she has not been able to optimally explore further concepts or respond to more complicated questions, such as explaining very small answers.

Verbal and verbal-symbolic approach

The subject YA solved problem A using a verbal-symbolic approach and problem B using a verbal approach. YA's test results are as displayed in Figure 9 and Figure 10.

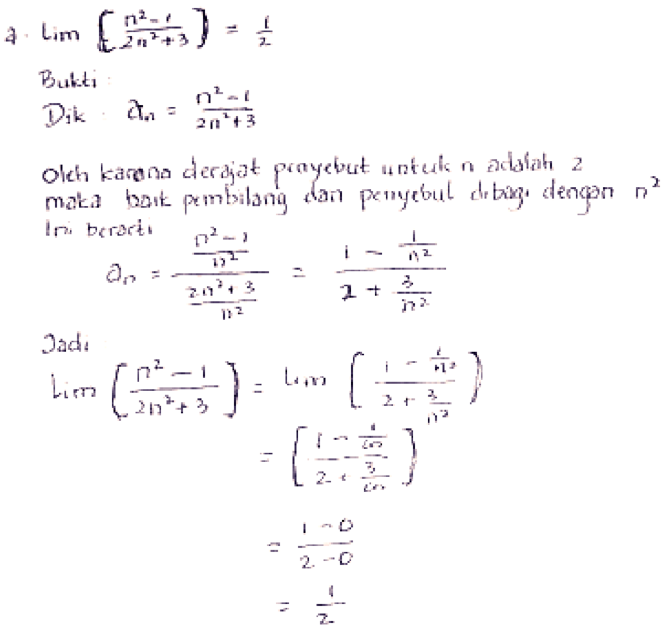

Figure 9. YA's test outcome for problem A

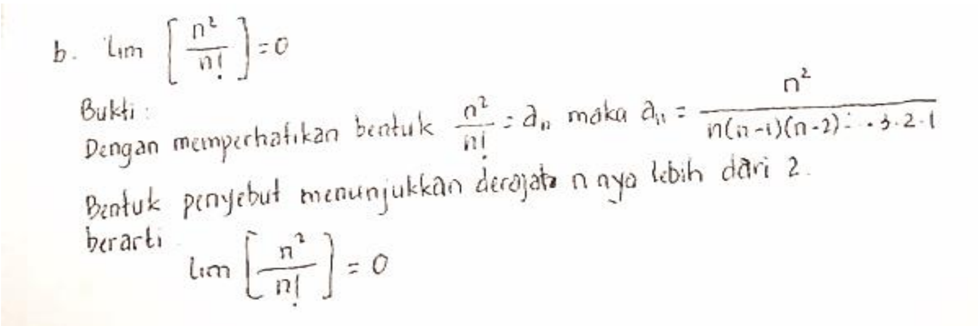

Figure 10. YA's test outcome for problem B

Based on the test results for problem A, YA integrates verbal-symbolic representations by conveying ideas and responding to problems that involve the use of mathematical symbols as well as manipulates and simplifies symbolic writing of the given mathematical expressions. Meanwhile, for problem B, YA solves the problem using verbal representations only, as also presented when solving problem A. This means that YA can communicate the concept of the limit of a sequence through the use of words clearly and effectively. These results are supported by an excerpt from an interview with YA about the ability to verbally and symbolically represent the limit of a sequence as follows:

> *"My idea for solving problem A was that I started paying attention to the form of a_n* [...] *where both the numerator and denominator contain n^2 and the highest degree of n in the denominator is 2, as is the numerator, Ma'am* [...] *as I previously understood* [...] *then, I divided the a_n in both the numerator and denominator by n^2. After that* [...] *a new form of a_n is obtained, namely $\left(1-\frac{1}{n^2}\right)$ per $\left(2+\frac{3}{n^2}\right)$ such that the limit (n^2-1) per $(2n^2+3)$ equals to the limit $\left(1-\frac{1}{n^2}\right)$ per $\left(2+\frac{3}{n^2}\right)$* [...] *The limit of this sequence is about n going towards infinity, Ma'am* [...] *so when substituted for n equals infinity, the result becomes $\left(1-\frac{1}{\infty}\right)$ per $\left(2+\frac{3}{\infty}\right)$* [...] *thus, the result is $(1-0)$ per $(2+0)$ equals $\frac{1}{2}$, Ma'am. Meanwhile, for problem B, Ma'am* [...] *actually, the idea is almost the same* [...] *it's just that here (while pointing at the answer sheet) I elaborated $n!$ into $n(n-1)(n-2)\ldots 3.2.1$. it's very clear that the denominator n has more degrees than the numerator whose n degree is only 2* [...] *that's why I immediately concluded that the limit is 0."*

Based on the interview, YA understands the relevant properties that can be applied so that it is clear how she communicates effectively in explaining the mathematical ideas involved in the solution. However, apart from the strengths exploited by YA, she also has weaknesses in the aspect of her inability to connect visual representations that support verbal-symbolic representations, as seen in the following excerpt from an interview with YA:

> *"I didn't try another way to solve these two problems, Ma'am* [...] *I'm still confused about how to illustrate the sequence a_n"*

Symbolic and visual approach

The subject EF solved problem A using a symbolic approach and problem B using a visual approach. EF's test results are displayed in Figure 11.

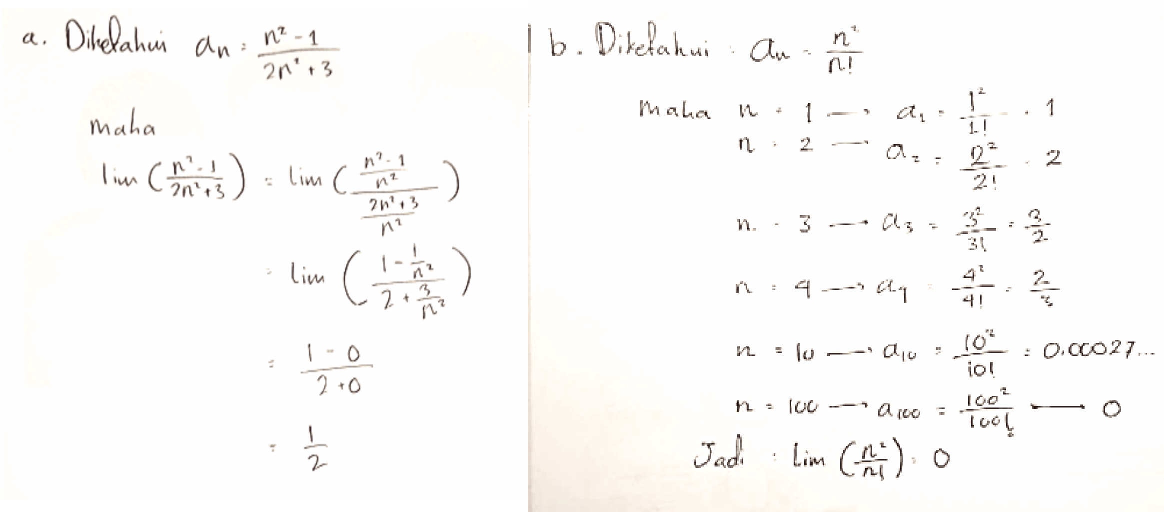

Figure 11. EF's test outcomes for problem A and B

Based on the test results, EF uses a different approach to solve each problem. EF can adapt to various types of problems and choose the most appropriate representation method for each situation. This skill is extremely important because some concepts or problems may be easier to understand or explain using one type of representation than another. For this matter, EF's answers were clarified through interview as follows:

> *"As for problem A* […] *I used the same method as in Calculus, Ma'am* […] *I divided both the numerator and denominator of* a_n *by* n^2*, Ma'am* […] *in the denominator, the highest degree of* n *is 2* […] *in Calculus, as far as I remember, the highest degree is considered. So, everything is divided by* n^2*, both the numerator and the denominator, Ma'am (while pointing at the answer sheet). The limit* $(n^2 - 1)$ *per* $(2n^2 + 3)$ *equals the limit* $\left(1 - \frac{1}{n^2}\right)$ *per* $\left(2 + \frac{3}{n^2}\right)$ […] *as the* n *value is getting bigger, the result becomes* $\left(1 - \frac{1}{\infty}\right)$ *per* $\left(2 + \frac{3}{\infty}\right)$ […] *thus the result is* $(1 - 0)$*per* $(2 + 0)$ *equals* $\frac{1}{2}$*. Meanwhile, for problem B* […] *I don't know how to apply the method I used in problem A to solve problem B* […] *although I think* $n!$ *has more degrees than* n^2 […] *but I'm still unsure of how to solve it using the same approach as in problem A, Ma'am. So the idea was to make a table, Ma'am* […] *because I didn't bring a ruler* […] *I made a list like that, Ma'am* […] *I inputted the* n *values one by one. In the beginning, it is not visible when inputting the* n *value from 1 to 4* […] *but when* n *equals 10, the value is already 0.00027 and so on* […] *then, for larger* n*, the value of* a_n *gets smaller and smaller towards 0, Ma'am* […] *That's why I concluded the limit is 0."*

Based on the interview, EF is able to convey understanding holistically by utilizing various forms of representation, thereby applying symbolic and sometimes visual approaches to solving the limit of sequence problems. This means EF has the ability to solve problems with a variety of approaches. However, apart from the strengths exploited by EF, she also has weaknesses in the aspect of difficulty in communicating problems clearly through images.

Apart from the dominant and non-dominant groups, some students were unable to solve the convergent sequence problems (see again Table 3). For example, HM and RM are subjects who could not represent the limit of a sequence for problem A; they both have trouble explaining or comprehending the concepts of the limit of a sequence. Both HM and RM are very quiet and difficult to communicate with in class, so they receive special assistance in mastering the materials. This case is not discussed further as it is outside the topic of this study.

Discussions

Each student has different preferences in representing the convergence of a sequence. Some students show a dominant tendency towards one particular type of representation based on their characteristics. This dominant representation is a representation ability that students always use to understand and solve problems.

The characteristics shown by each student are in line with the results of several previous studies. In the case of the subject who has dominant verbal representation, HP can present the problem clearly using easy-to-understand words. The conveyance of concepts is also done by detailing ideas and arranging them using logical reasoning and thinking. This is in line with the result of a previous study which shows that students who have verbal representations tend to have the ability to develop their thinking through presenting verbal arguments or translating the nature of the problem and the relationships found therein into verbal expressions (Demircioglu et al., 2023; Mainali, 2021). Nevertheless, HP is weak in abstract concepts in the aspects of problem visualization and the use of symbols or algebraic forms. This contradicts a prior study that reported the importance of having the ability to explore visualization of a problem as a support for how students learn to connect their understanding with their knowledge, i.e., their representation (Ramírez-Uclés & Ruiz-Hidalgo, 2022).

As the subject with a dominant visual representation, IS can explore visual representations by using table and image approaches to relate existing information (sequence patterns). This supports the findings of previous studies that students who have visual representations have the ability to interpret or translate visual displays in the problem-solving process (Parame-Decin, 2023; Utami et al., 2021; Zorzos & Avgerinos, 2023). However, IS's visual representation is unelaborate, less formal, and does not have much mathematical notation, which can lead to ambiguity, errors in understanding concepts, and difficulties in translation (Mainali, 2021; Cooper et al., 2018). Therefore, it is crucial to strike a balance between simplicity and clarity in the illustrations provided, especially when dealing with complex mathematical concepts.

TR, the subject with dominant symbolic representation, has the ability to formulate arguments formally (by definition) as seen from her ability to effectively use notation to explain concepts in proving the limit of a sequence. Therefore, TR is quite confident in conveying complex ideas precisely and consistently. According to lecturers who teach real analysis courses, this condition is anticipated and ideal for students to have when studying the convergence of a sequence. Students are expected to have the ability to demonstrate mathematical situations and recognize the structure and meaning in symbolic expressions (Mutammam & Wulandari, 2023; Mutodi & Mosimege, 2021). However, TR experienced problems when asked to present her ideas geometrically. This is partly due to a lack of experience in linking symbolic and visual representations of a concept (Ebissa, 2020; Žakelj & Klančar, 2022).

The findings about the characteristics of subjects who are dominant in one type of mathematical representation confirm that subjects' representation preferences can help in designing communication approaches in learning to make them more effective. However, educators should direct students to develop their understanding of other representations in order to train them to solve more complex problems (Fries et al., 2021; Hadiastuti et al., 2019; Mainali, 2021; Ratumanan et al., 2022).

For subjects who are not dominant in one type of representation, on the other hand, their characteristics show diversity which reflects flexibility in understanding the convergence of a sequence. This finding is in line with previous studies which reveal that students with flexible understanding have the opportunity to be able to solve several problems with various forms of representation, depending on their experience and knowledge (Afriyani et al., 2018; Darling-Hammond et al., 2020; Hadiastuti et al., 2019; Mainali, 2021;).

In relation to learning convergent sequences, visual, verbal and symbolic representation abilities certainly help students discover ways to communicate ideas to find answers and direct mathematical thinking (L.Man et al., 2022; Mainali, 2021; Himmah & Rahaju, 2021). Furthermore, students can have a more comprehensive learning experience when they can build and interpret diverse representation models according to their own characteristics (Fatmawati et al., 2022).

By considering alternative learning approaches, profound insight can be provided to students. This is achieved through the ability to relate concepts to various life contexts and understand their implications more comprehensively by involving skills in presenting evidence formally and mathematically, thus demonstrating a high level of accuracy both in writing and understanding the solution steps.

Conclusions

In finding the limit of a sequence, the characteristics of the representation used by students play a crucial role. This representation hones students' skills in presenting each solution step with accuracy and consideration of each assumption made. The ability to use representations, both dominant and non-dominant, is expected to help students think critically and analytically. Through this activity, students are motivated to explain the steps taken and why these steps are necessary to reach a conclusion. By choosing the right representation, students can clearly describe how a problem is solved. In addition, they can effectively communicate their mathematical understanding to others by considering the assumptions made and explaining the steps clearly. This study found that the dominant characteristics of each student are reflected in the form of representation chosen or in their unique way of understanding and solving problems. For example, students who often use verbal representations tend to solve problems by developing their thoughts in words or changing the properties and relationships observed in the problem into verbal expressions. Students with dominant visual representation are likely to solve problems by interpreting or using visual displays, while those with dominant symbolic representation have the tendency to solve problems by breaking them down into several steps and developing their understanding conceptually. Furthermore, this study reveals that students with no dominant representation have various characteristics reflecting flexibility and understanding that allow them to solve problems using a variety of representations. Therefore, further studies are absolutely needed to discover more about solution patterns based on activity flow in representing a problem as an alternative learning approach as to strengthen the findings of this study.

References

Adeoye, M. A., & Jimoh, H. A. (2023). Problem-solving skills among 21st-century learners toward creativity and innovation ideas. *Thinking Skills and Creativity Journal*, *6*(1), 52–58. https://doi.org/10.23887/tscj.v6i1.62708

Afriyani, D., Sa'dijah, C., Subanji, S., & Muksar, M. (2018). Characteristics of students' mathematical understanding in solving multiple representation task based on solo taxonomy. *International Electronic Journal of Mathematics Education (IEJME)*, *13*(3), 281–287. https://doi.org/10.12973/iejme/3920

Angraini, L. M., & Wahyuni, A. (2021). The effect of concept attainment model on mathematical critical thinking ability. *International Journal of Instruction*, *14*(1), 727–742. https://doi.org/10.29333/iji.2021.14144a

Arnal-Palacián, M., Claros-Mellado, J., & Sánchez-Compaña, M. T. (2020). Infinite limit of sequences and its phenomenology. *International Electronic Journal of Mathematics Education (IEJME)*, *15*(3), 1–13. https://doi.org/10.29333/iejme/8279

Bartle, R. G., & Sherbert, D. R. (2011). Introduction to real analysis. In *University of Illinois, Urbana-Champaign* (4th ed.). John Wiley & Sons, Inc.

Çelik, H. C., & Özdemir, F. (2020). Mathematical thinking as a predictor of critical thinking dispositions of pre-service mathematics teachers. *International Journal of Progressive Education*, *16*(4), 81–98. https://doi.org/10.29329/ijpe.2020.268.6

Cooper, J. L., Sidney, P. G., & Alibali, M. W. (2018). Who benefits from diagrams and illustrations in math problems ? Ability and attitudes matter. *Applied Cognitive Psychology*, *38*, 24–38. https://doi.org/10.1002/acp.3371

Creswell, J. W., & Creswell, J. D. (2018). *Research design qualitative, quantitative, and mixed methods approaches* (5th ed.). SAGE Publications, Inc.

Darling-Hammond, L., Flook, L., Cook-Harvey, C., Barron, B., & Osher, D. (2020). Implications for educational practice of the science of learning and development. *Applied Developmental Science*, *24*(2), 97–140. https://doi.org/10.1080/10888691.2018.1537791

Demircioglu, T., Karakus, M., & Ucar, S. (2023). Developing students' critical thinking skills and argumentation abilities through augmented reality–based argumentation activities in science classes. In *Science and Education* (Vol. 32, Issue 4). Springer Netherlands. https://doi.org/10.1007/s11191-022-00369-5

Diarni, I. M., Ikhsan, M., Zaura, B., Johar, R., & Mailizar, M. (2023). Profile of students' mathematical understanding through diagnostic tests viewed from multiple intelligences. *Jurnal Didaktik Matematika*, *10*(1), 77–92. https://doi.org/10.24815/jdm.v10i1.31965

Ebissa, L. (2020). Improving geometric concepts perceived difficult to first year linear mathematics students ' of Kemissie College of Teachers Education in 2019 G . C . *International Journal of Creative Research Thoughts (IJCRT)*, *8*(11), 276–288.

Fatmawati, A., Zubaidah, S., Mahanal, S., & Sutopo, S. (2022). Representation skills of students with different ability levels when learning using the LCMR model. *Pegem Journal of Education and Instruction*, *13*(1), 177–192. https://doi.org/10.47750/pegegog.13.01.20

Feriyanto. (2017). The ability of students' mathematical proof in determining the validity of argument reviewed from gender differences. *Journal of Physics: Conference Series*, *947*(1), 1–6. https://doi.org/10.1088/1742-6596/947/1/012042

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to Design and Evaluate Research in Education* (8th ed.). McGraw-Hill.

Fries, L., Son, J. Y., Givvin, K. B., & Stigler, J. W. (2021). Practicing connections : a framework to guide instructional design for developing understanding in complex domains. *Educational Psychology Review*, *33*, 739–762. https://doi.org/10.1007/s10648-020-09561-x

Hadiastuti, D. I., Soedjoko, E., & Universitas, M. (2019). Analysis of mathematical representation ability based on students' thinking style in solving open-ended problems. *Unnes Journal of Mathematics Education*, *8*(3), 195–201. https://doi.org/10.15294/ ujme. v8i3.34189

Hariyani, M., Suherman, S., Andriani, M., & Herawati, H. (2023). The Importance of mathematical representation ability for elementary school students: a literature review and its implications. *Syekh Nurjati International Conference on Elementary Education*, *1*(0), 38–46.

https://doi.org/10.24235/sicee.v1i0.14579

Herizal, H., Suhendra, S., & Nurlaelah, E. (2019). The ability of senior high school students in comprehending mathematical proofs. *Journal of Physics: Conference Series*, *1157*(2), 2–7. https://doi.org/10.1088/1742-6596/1157/2/022123

Himmah, M., & Rahaju, E. B. (2021). Analysis of student's mathematics representation in solving mathematics problems based on spatial cognitive style. *MATHEdunesa*, *10*(2), 189–199. https://doi.org/10.26740/mathedunesa.v10n2.ppdf_189-199

Isnani, Waluya, S. B., Rochmad, Dwiyanto, & Asih, T. S. N. (2021). Analysis of problem-solving difficulties at limits of sequences. *Journal of Physics: Conference Series*, *1722*(1). https://doi.org/10.1088/1742-6596/1722/1/012033

Iwuanyanwu, P. N. (2022). What students gain by learning through argumentation. *International Journal of Teaching and Learning in Higher Education*, *34*(1), 97–107. http://www.isetl.org/ijtlhe/

Kola, M., & Molise, H. (2023). Assessing the implementation of critical thinking skills in the university: a focus on technology education. *E-Journal of Humanities, Arts and Social Sciences*, *4*(5), 500–515. https://doi.org/10.38159/ehass.2023451

Kristianto, E., Mardiyana, & Saputro, D. R. S. (2019). Analysis of students' error in proving convergent sequence using newman error analysis procedure. *JIOP Conf. Series: Journal of Physics: Conf. Series*, *1180*(012001), 1–7. https://doi.org/10.1088/1742-6596/1180/1/012001

Kumar Shah, R. (2019). Effective constructivist teaching learning in the classroom. *Shanlax International Journal of Education*, *7*(4), 1–13. https://doi.org/10.34293/education.v7i4.600

L.Man, Y., Asikin, M., & Sugiman. (2022). Systematic literature review: student's mathematical representation ability in mathematics learning. *Daya Matematis : Jurnal Inovasi Pendidikan Matematika*, *10*(1), 36–44. https://doi.org/10.26858/jdm.v10i1.26821

Langoban, M. A. (2020). What makes mathematics difficult as a subject for most students in higher education? *International Journal of English and Education*, *9*(3), 214–220.

Lima, P. D. S. N., Silva, L. D. A., Felix, I. M., & Brandao, L. D. O. (2019). Difficulties in basic concepts of mathematics in higher education: a systematic review. *Proceedings - Frontiers in Education Conference, FIE*, *2019-Octob*(November). https://doi.org/10.1109/FIE43999.2019.9028658

Mainali, B. (2021). Representation in teaching and learning mathematics. *International Journal of Education in Mathematics, Science and Technology*, *9*(1), 1–21. https://doi.org/10.46328/ijemst.1111

Maslihah, S., Waluya, S. B., Rochmad, R., & Suyitno, A. (2020). The role of mathematical literacy to improve high order thinking skills. *Journal of Physics: Conference Series*, *1539*(1), 1–6. https://doi.org/10.1088/1742-6596/1539/1/012085

Munna, A. S., & Kalam, M. A. (2021). Teaching and learning process to enhance teaching effectiveness: literature review. *International Journal of Humanities and Innovation (IJHI)*, *4*(1), 1–4. https://doi.org/10.33750/ijhi.v4i1.102

Mutammam, M. B., & Wulandari, E. N. (2023). Profile of junior high school students' symbol sense thinking. *JTAM (Jurnal Teori Dan Aplikasi Matematika)*, *7*(2), 509–521. https://doi.org/10.31764/jtam.v7i2.12622

Mutodi, P., & Mosimege, M. (2021). Learning mathematical symbolization: conceptual challenges and instructional strategies in secondary schools. *Bolema - Mathematics Education Bulletin*, *35*(70), 1180–1199. https://doi.org/10.1590/1980-4415v35n70a29

Novianti, M., & Retnawati, H. (2019). Student-teacher's perception of mathematical representation in mathematics learning. *Journal of Physics: Conference Series*, *1320*(1), 1–6. https://doi.org/10.1088/1742-6596/1320/1/012106

Nurdin, N., Assagaf, S. F., & Arwadi, F. (2021). Students' understanding on formal definition of limit. *Journal of Physics: Conference Series*, *1752*(012082), 1–4. https://doi.org/10.1088/1742-6596/1752/1/012082

Parame-Decin, M. B. (2023). Visual representations in teaching mathematics. *Sprin Journal of Arts, Humanities and Social Sciences*, *2*(05), 21–30. https://doi.org/10.55559/sjahss.v2i05.107

Peňalber, M. D. (2023). The practice of gardner's multiple intelligences theory in the classroom. *Journal for Educators, Teachers and Trainers*, *14*(4), 62–74. https://doi.org/10.47750/jett.2023.14.04.006

Ponnusamy, S. (2012). Sequences: Convergence and Divergence. In *Foundations of mathematical analysis* (pp. 23–71). Springer Science+Business Media. https://doi.org/10.1007/978-0-8176-8292-7

Prayitno, S., Lu'luilmaknunn, U., Sridana, N., & Subarinah, S. (2021). Analyzing the ability of mathematics students as prospective mathematics teachers on multiple mathematical representation.

Proceedings of the 2nd Annual Conference on Education and Social Science (ACCESS), *556*, 309–313. https://doi.org/10.2991/assehr.k.210525.096

Rabih, M. N. A. (2017). On convergence criteria for sequences. *IJRR International Journal of Research & Review*, *4*(5), 87–91. https://doi.org/10.4444/ijrr.1002/375

Ramírez-Uclés, R., & Ruiz-Hidalgo, J. F. (2022). Reasoning, representing, and generalizing in geometric proof problems among 8th grade talented students. *Mathematics*, *10*(789), 1–21. https://doi.org/10.3390/math10050789

Rashid, Y., Rashid, A., Warraich, M. A., Sabir, S. S., & Waseem, A. (2019). Case study method: a step-by-step guide for business researchers. *International Journal of Qualitative Methods*, *18*, 1–13. https://doi.org/10.1177/1609406919862424

Ratumanan, T. G., Ayal, C. S., & Tupamahu, P. Z. (2022). Mathematical representation ability of mathematics education study program students. *Jurnal Pendidikan Matematika (JUPITEK)*, *5*(1), 50–59. https://doi.org/10.30598/jupitekvol5iss1pp50-59

Rosa, M., D'Ambrosio, Ubiratan; Orey, D. C., Shirley, L., Alangui, W. V., & Palhares, Pedro; Gavarrete, M. E. (2016). *Current and future perspectives of ethnomathematics as a program* (ICME-13 To). Springer Open. https://doi.org/10.1007/978-3-319-30120-4_1

Ross, K. A. (2013). *Elementary analysis : the theory of calculus* (2nd ed.). Springer Science+Business Media. https://doi.org/10.1016/S0049-237X(09)70531-2

Schüler-Meyer, A. (2020). Mathematical routines in transition: facilitating students' defining and proving of sequence convergence. *Teaching Mathematics and Its Applications: An International Journal of the IMA*, *39*(Januari), 237–247. https://doi.org/10.1093/teamat/hrz019

Sebsibe, A. S., & Feza, N. N. (2020). Assessment of students ' conceptual knowledge in limit of functions. *International Electronic Journal of Mathematics Education*, *15*(2), 1–15. https://doi.org/10.29333/iejme/6294

Subedi, A. (2020). Graduate level students' techniques and difficulties in proving theorems of abstract algebra. *Education and Development*, *30*(1), 99–112. https://doi.org/10.3126/ed.v30i1.49515

Tuna, A., Biber, A. C., & Korkmaz, S. (2019). What do teacher candidates know about the limits of the sequences? *Journal of Curriculum and Teaching*, *8*(3), 132–142. https://doi.org/10.5430/jct.v8n3p132

Utami, A. P., Mardiyana, & Pramudya, I. (2021). Visual students: how their representation in problem solving? *Proceedings of the International Conference of Mathematics and Mathematics Education (I-CMME 2021)*, *597*, 31–41. https://doi.org/10.2991/assehr.k.211122.005

Utomo, D. P., & Syarifah, D. L. (2021). Examining mathematical representation to solve problems in trends in mathematics and science study: voices from indonesian secondary school students. *International Journal of Education in Mathematics, Science and Technology*, *9*(3), 540–556. https://doi.org/10.46328/IJEMST.1685

Vintere, A. (2018). A constructivist approach to the teaching of mathematics to boost competences needed for sustainable development. *Rural Sustainability Research*, *39*(334), 2–7. https://doi.org/10.2478/plua-2018-0001

Widiati, I., & Sthephani, A. (2018). Difficulties analysis of mathematics education students on the real analysis subject. *The 6th South East Asia Design Research International Conference (6th SEA-DR IC)*, *1088*, 1–5. https://doi.org/10.1088/1742-6596/1088/1/012037

Žakelj, A., & Klančar, A. (2022). The role of visual representations in geometry learning. *European Journal of Educational Research*, *11*(3), 1393–1411. https://doi.org/10.12973/eu-jer.11.3.1393

Zorzos, M., & Avgerinos, E. (2023). Research on visualization in probability problem solving. *Eurasia Journal of Mathematics, Science and Technology Education*, *19*(4), 1–10. https://doi.org/10.29333/EJMSTE/13066