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Theoretical perspectives

Procedural knowledge or conceptual knowledge? Developing the so-called proceptual knowledge in mathematics learning

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Abstrak Sebagian siswa mungkin memiliki pengetahuan yang tepat dalam menggunakan prosedur matematika secara relevan, namun apakah mereka sungguh memiliki pemahaman yang utuh tentang "mengapa atau bagaimana" prosedur matematika tersebut diperoleh? Pemahaman yang tidak utuh tersebut berpotensi menjadi penghalang kesuksesan siswa dalam memahami konsep matematika. Artikel ini mengusulkan kerangka kerja untuk mengembangkan *proceptual knowledge* (pengetahuan proseptual) pada materi turunan, yakni kombinasi pengembangan pengetahuan prosedural dan konseptual matematika yang dibangun di atas teori-teori pembelajaran matematika yang ada, serta hasil refleksi pribadi penulis dari proses belajar mandiri tentang konsep rumus diferensial. Perspektif teoritis dan praktis yang diusulkan dalam artikel ini dapat menjadi panduan bagi siapa saja untuk mengembangkan pengalaman belajar matematika yang lebih bermakna, khususnya pada topik dengan rumus dan prosedur matematis yang kompleks seperti pada turunan.

Kata kunci *Pengetahuan proseptual, Pengetahuan prosedural, Pengetahuan konseptual, Turunan*

Abstract Some students might have the proper knowledge to use mathematical procedures where relevant, but do they actually have a solid understanding of "why or how" those procedures work? Such an incomplete understanding of mathematics concepts can be a stumbling block in students' success in mathematics. This paper aims to propose and elaborate a framework for developing proceptual knowledge combining both procedural and conceptual knowledge on differentiation that are constructed on existing mathematics learning theories on how we understand mathematics, besides my personal reflections from the independent learning on differentiation. The theoretical and practical perspectives proposed in this article share insight with anyone in developing a more meaningful mathematics-independent learning experience, especially on topics with complex mathematical formulas or procedures, such as differentiation.

Keywords *Proceptual knowledge, Procedural knowledge, Conceptual knowledge, Differentiation*

Introduction

It was still stuck in my mind the moment when I could not properly answer my sister's query about the derivatives of trigonometry in calculus. "How can the derivative of sinus equal to cosines, whereas the derivative of cosines equals negative sinus?" she asked. All that I could say was, "well, it is true, but that is the way it is. It is just some basic rules of differentiation that you have to remember." In spite of the fact that I was aware of those differentiation formulae, I

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had yet to learn why and how those equations worked. The response I gave to my sister arguably is the typical answer we commonly receive when asking why and how certain formulae in mathematics work. It was considered the same when our maths teachers said, "turn it upside down and then multiply" for the division of fractions. At this point, it can be indicated that mathematics concepts were offered as ready-made products, which have been that way from the start (Li & Schoenfeld, 2019). Hence, students need to use the fixed formula or strategy in solving any mathematics problems without having to make sense of the formula.

Differentiation was one topic area in mathematics that I found personally very challenging back in secondary school, as it contains many symbols and complex formulae that were merely learned through drill and memorisation. I was not the only one who struggled with this topic area since many pieces of literature had confirmed students' difficulties in grasping the ideas behind the concept of derivatives in calculus (Burns, 2014; Hashemi et al., 2015; Maharaj, 2013; Tall, 1992). Some students could remember some of the differential equations but likely not be able to provide proper answers when confronted with a question that tested their logical rationalisation of the formulae. I am going to depict what I mean from my previous assertion through this following conversation with my lecturer during the calculus course I took back in my undergraduate degree. L stands for the lecturer, while M stands for my initial name.

L : Do you know what the derivative of $y=x^2$ *is?*
M : Yes. It is 2x.
L : How do you get the result?
M : As I know, in differentiation of any exponential function, whatever the exponent upstairs, it moves in front of the constant number, and we take one away from the exponent.
L : Why do you move its exponent to the front and then take one away from it?
M : I don't know. That's the rule that I remember when I learned derivatives at school.

The dialogue above clearly demonstrates the fact that I know the derivative rules and have sufficient ability to use them. However, at the same time, it implies that I just know "what" the rules involved but with a very limited understanding of "why" it works that way. Skemp (2020) referred to such a piece of knowledge as instrumental understanding, that is, "knowing rules without reasons". In other words, this assertion implies that memorisation of mathematics rules often comes without understanding. While this kind of knowledge does not necessarily always bring detrimental impacts on students' development in mathematics -which will be discussed further in the next section-, focusing solely on developing this kind of knowledge will lead to short-term memory (Ferlazzo, 2020) and inflexibility in adopting the procedures into new problems (Skemp, 2020). That is to say, students with merely procedural knowledge might encounter trouble when given a mathematics problem that does not quite fit the procedures they had learned. Some empirical studies have confirmed that students who learned mathematics solely in a procedural-based approach tend to develop an inert knowledge that was of limited use to them (Arslan, 2010; Boaler, 1998). For this reason, therefore, it is crucial to review the learning theory behind developing knowledge in mathematics. Through the review, I intend to dig a more in-depth understanding of derivatives by focusing on rationalising all the differential equations through self-study. However, before discussing further how I develop my knowledge of derivatives, some theoretical perspectives and technical terms used in this piece of work will be initially defined and discussed in the following section.

Theoretical Review

Different types of knowledge in mathematics

Anderson and Krathwohl (2001) stated that knowledge is what is known or expected to be known by learners, which is gained through a cognitive process. "Knowledge" and "the way learners understand" are basically two closely related yet distinct concepts. The latter indicates the action of grasping meaning or an active process of knowing (Sierpinska, 1990) and a continuous development of cumulating knowledge (Michener, 1978). From these definitions, it can be argued that knowledge is a "product" of the cognitive process, while the way someone understands refers to the "process" of cognition. In the context of mathematics, there are two fundamental types of knowledge that students gain from mathematics learning. The first one is knowledge in carrying out a number of mathematical problems using skills, algorithms, methods, and procedures, whereas the other one is knowledge in having a 'sense' of the mathematical concepts and skills (e.g. Canobi, 2009; Hiebert & Lefevre, 1986; Hurrell, 2021; Long, 2005; Miller & Hudson, 2007; Rittle-Johnson, 2017; Skemp, 2020). Interestingly, those two distinct types of knowledge are technically termed differently across the literature. Many pieces of literature termed the former as procedural knowledge, while the latter was known as conceptual knowledge (Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999). Meanwhile, Baroody and Ginsburg (1986) distinguished the two types of knowledge as mechanical and meaningful knowledge, whereas Skemp (1976) characterised the two as instrumental understanding and relational understanding consecutively. Despite those different labels across literature, each of the terminologies refers to considerably the same definition (Hiebert & Lefevre, 1986).

Then, the question might arise whether or not one piece of knowledge is much better than the other. Skemp (1976) argued that knowledge of the procedure, which he referred to as instrumental understanding, does not indicate understanding at all if it merely knows the rule and how to use it. This, in my interpretation, is closely linked to the emphasis on memorisation or habitual repetition of any mathematical rules as in rote learning, which, as per Long (2005), does not create meaningful knowledge or skills. Besides, students who learn solely in a procedural way tend to do mathematics according to a set of mathematical rules that mainly involves memorisation with no or very limited understanding of the underlying meanings of those rules (Arslan, 2010). As a consequence, students can end up with peculiar and unreasonable solutions (Martin, 2009), and each mathematical concept that is learned appears to be fragmented and has no relation to the other concepts (Li & Schoenfeld, 2019). Despite all those drawbacks, this does not mean that knowledge of procedures harms students' learning. Skemp (2020) argued that students could benefit from such knowledge and provoke three possible advantages. First, some mathematics concepts are much easier and quicker to understand through procedural ways, such as "turn it upside down and multiply for division by a fraction" (p.8). Secondly, as students can get the right answers using procedural ways, the feeling of success and the rewards are more immediate they get. Lastly, it involves less knowledge yet is still reliable.

On the contrary, conceptual knowledge indicates the flexibility of the knower in manipulating mathematical symbols in mind (see Gray & Tall, 1994) and in understanding the mathematical concepts and also indicates an understanding of the interrelations among those concepts (Rittle-Johnson & Alibali, 1999). The interrelationship can be between two or more mathematical concepts or between a concept previously learned and the newly learned (Rittle-

Johnson et al., 2016). Such flexible thoughts mean an ability to adapt and change methods to fit new problem situations (Skemp, 2020), and therefore it profoundly contributes to students' success and performance when working on mathematics problems (Boaler, 1993).

Proceptual knowledge

Students typically go through a procedurally oriented phase before they can effectively integrate their conceptual knowledge (Tall et al., 1999). In spite of the fact that those who solely concentrate on the procedure can be very good at computations and succeed in the short term, they may lack the flexibility that will give them ultimate success in the long term (Gray & Tall, 1992). However, this does not necessarily mean that students would always devastate by procedural knowledge because even relational mathematicians often use instrumental thinking (Skemp, 1976).

As a matter of fact, procedural fluency and conceptual understanding are two out of five predominant strands of mathematical fluency that indicate someone understands and can do mathematics (National Research Council, 2001). Despite the long-standing debate about which type of knowledge is better acquired by pupils, the previous assertion explicitly indicates the significance of the two types of knowledge. Both procedural and conceptual knowledge are critically important and powerful to be developed in understanding mathematics and help to strengthen each other et al., 2015). Long (2005) asserted that conceptual knowledge is closely linked to procedural knowledge, where knowledge of procedures is nested in conceptual knowledge. Rittle-Johnson and Schneider (2015) supported the idea of an inseparable linkage between procedural and conceptual knowledge and argued that the two constructs are bi-directional; that is to say, procedural knowledge supports conceptual knowledge, and vice versa.

Gray and Tall (1992) proposed the idea of *procept*, which is a combination of mathematical processes and concepts. With this notion in mind, mathematics learning should encourage pupils to have the proper knowledge to use mathematical procedures where relevant and give meaning to the process in a flexible way that allows process and concept to be interchanged at will, often without any distinction being made between the two. As having been alluded to at the beginning about instrumental understanding, the opposite category of this understanding is relational understanding (Skemp, 2020), that is, “knowing both what to do and why”. Interestingly, different ways of understanding mathematics will contribute to the acquisition of different knowledge (Lampert, 1986). That is to say, a student who understands mathematics instrumentally will likely merely gain procedural knowledge, while on the other hand, learning which emphasises developing learners’ relational understanding will provide the learners with proceptual knowledge.

The Proposed Framework

Developing proceptual knowledge on derivatives

I was initially confused about how to start developing my knowledge of derivatives. What I did at the very beginning of my learning process was trying to solve a number of problems related to differentiation. I was doing this because I had no longer practised for the last few years, and I would also train my memory to solve more complex derivative problems. I could solve straightforward problems that require simple memory, such as solving problems using the power rule, e.g., formulae 1 to 5 in Figure 1. However, I encountered difficulties when attempting problems involving quotient or product rule of derivatives, e.g., formulae 6 and 7 in Figure 1.

One plausible reason that could explain such difficulties is because rule 6 and 7 of Figure 1 involve mathematical steps or procedures that I had no idea how they worked.

1. $y = c$	$\Rightarrow y' = 0$
2. $y = x^n$	$\Rightarrow y' = nx^{n-1}$
3. $y = ax^n$	$\Rightarrow y' = anx^{n-1}$
4. $y = U + V$	$\Rightarrow y' = U' + V'$
5. $y = U - V$	$\Rightarrow y' = U' - V'$
6. $y = U . V$	$\Rightarrow y' = U'V + UV'$
7. $y = \frac{U}{V}$	$\Rightarrow y' = \frac{U'V - UV'}{V^2}$

Figure 1. Some derivatives formulae I had to memorise back in school

Therefore, I tried to put myself as a high school learner who had just started learning differentiation. I was helped by the approach described by Coles (2016) in his book "*Engaging in Mathematics in the Classroom*." Instead of directly offering learners ready-made formulae of the differential equation, the idea of differentiation is introduced by involving the measurement of a gradient of a graph. I learned that gradient basically represents the slope of the tangent of a graph of a function, like the differentiation. An example of finding the gradient of $y=x^2$ at $x=1$ was given in that book. I did the same numerical process for $y=x^2+x$ from $x=1$, as shown in Figure 2, to several points in the x-coordinate, such as $x=2$; $1\frac{1}{2}$, and $1\frac{1}{3}$. At this starting point, using graphs helped me to visualize what the concept of differentiation looks like.

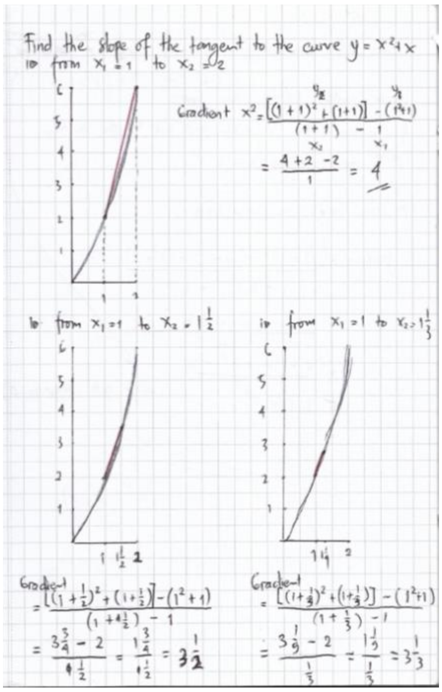

Figure 2. Finding the slope of the tangent line of the curve $y=x^2+x$

Having seen the gradients above, I got 4, $3\frac{1}{2}$, and $3\frac{1}{3}$. I could spot the pattern of the slopes here, and therefore, I could predict the gradient of the tangent to various curves at *x=1*, such as $3\frac{1}{4}$, as the slope of the line from *x=1* to *x=1*$\frac{1}{4}$. It can be clearly seen that we will get an infinite sequence of slopes of lines as the slope tends to the tangent (Cohen, 1991). In general, if I extend this sequence of gradient and call *h* the change in *x*-coordinate (the number I add to the abscissa from *x=1*), I got

$$Gradient = \frac{[(1+h)^2 + (1+h)] - (1^2+1)}{(1+h) - 1}$$

As *h* gets closer and closer to 0, the limit of this sequence can be written as:

$$Gradient = \lim_{h \to 0} \frac{[(1+h)^2 + (1+h)] - (1^2+1)}{(1+h) - 1}$$

After expanding the brackets and simplifying, I get the limit of this sequence:

$$Gradient = \lim_{h \to 0} 3 + h = 3$$

I did the same activity to find the slope of the tangent to the curve $y = x^2 + x$ at various points (i.e. $x = 2$ and $x = 3$, and I could predict the gradient of $y = x^2 + x$ at $x = 4$ or 5 and so forth:

X-coordinate	1	2	3	4	5
The gradient of the tangent	3	5	7	9	11

At this point, I arrived at the general pattern of the gradient of the tangent to the curve $y = x^2 + x$ as follows.

$$\begin{aligned} Gradient &= \lim_{h \to 0} \frac{[(x+h)^2 + (x+h)] - (x^2+x)}{(x+h) - x} \\ &= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) + (x+h) - (x^2+x)}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + h}{h} \\ &= \lim_{h \to 0} 2x + h + 1 \\ &= 2x + 1 \end{aligned}$$

Having been working on this numerical process, I have arrived at the definition of the differential of a function.

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Definition of Differentiation: The limit of the ratio of the change of a function (*y*-ordinate) to the change of a variable in it (*x*-axis) as the latter limits to zero (Cohen, 1991). Accordingly, I learned that gradient is the first derivative of a function. From this definition, therefore, I started to prove most of the differential equations that I learned in secondary school in order to rationalise all those equations and overcome my lack of understanding of how those derivatives formulae are obtained. Figure 3 is some snapshots from my notebook during the independent learning I carried out in an effort to comprehend and to prove each formula of derivatives which I used to have to memorise yet without understanding.

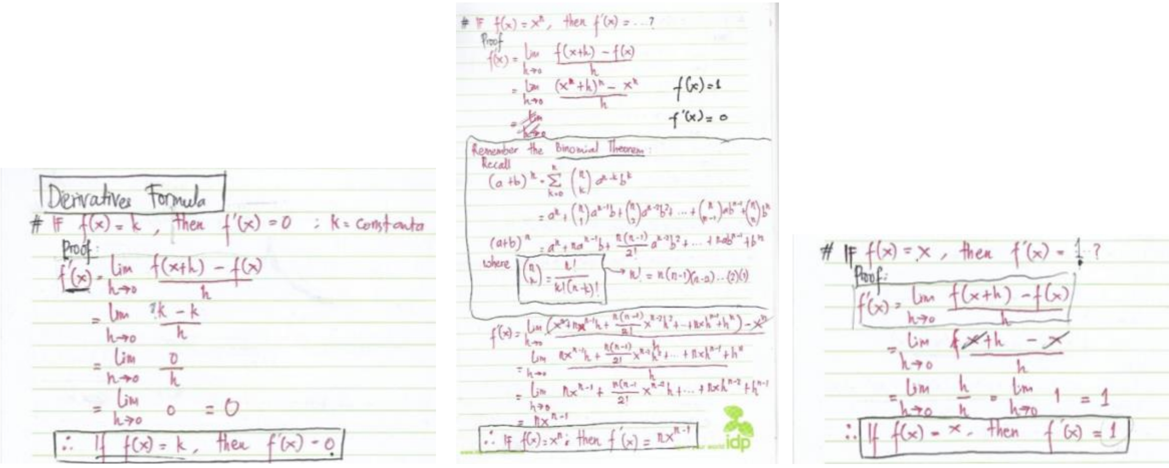

Figure 3. Some snapshots from the self-study in proving the derivatives formulae

Reflection from the learning process

I began my independent study by directly attempting some mathematics problems related to derivatives using the formulae that I could remember. However, I noticed that new knowledge had not been acquired or developed through this activity because I did just drill using the basic formulae already rooted in my memory. On the other hand, when solving some more complex problems, such as finding the derivatives of multiplication and/or division of two functions, I got to reopen my book to get the formulae involved because I already forgot them.

At this point, where I still relied heavily on the various formulae of differentiation, I construe that my knowledge of differentiation had not been well internalised yet. One possible explanation for that is the fact that I did not comprehend what the symbols in the differential formulae mean and where they are derived from. Moreover, I had to recall different formulae of differentiation to solve different problems; that is to say, the way I learned derivatives in school unconsciously drove me to the conclusion that there is no interrelationship among the formulae, which resulted in some formulae I have forgotten. This short-term memory of differentiation becomes obvious evidence of a result of procedural understanding embedded in my learning by which memorisation of rules was dominating, but a lack of reasoning on those rules was evident (Skemp, 2020). Such an overemphasis on memorising activity in learning might be the reasonable factor that drives me to the point of "memorisation overload" (Cornell, 1999), so that while new knowledge is acquired, the rarely trained knowledge might perish.

On the contrary, once I changed my learning approach by first trying to comprehend the definition of derivatives using the measurement of the gradient of a graph (Cohen, 1991; Coles, 2016), I could gradually understand the underlying concepts of derivatives and ultimately arrived at the point where I identify myself to develop a more conceptual knowledge rather than the procedural one. As an illustration, I initially did not know that gradient and the concept of limit have a relationship to derivatives since I used to directly employ the ready-made formulae of derivatives in solving problems that do not require computation involving the concepts of limit. Furthermore, having understood the definition of differentiation and its intertwinement with other mathematical concepts, I have been able to rationalise the formulae or rules of derivative: how those rules are derived from, and also to solve a number of differential problems using this definition. If in case I forgot any formulae involved in solving differential problems, I could just recall the definition of differentiation that has been rooted in my memory. Understanding the concept of derivatives 'relationally' benefits me in the sense that less memory work is involved, and what has to be memorised is in the interrelated form that can be easily retained in my mind (Skemp, 2020).

Apart from the aforementioned arguments, it does not necessarily mean that the role of shortcuts like memorising the derivative formulae should be completely avoided in learning. This is because when attempting to derive problems solely using the definition, one might still be vulnerable to making computational errors. Figure 4 illustrates a mistake due to a computational mistake when I tried to find the derived value of a function using the definition of the derivative. I wrote in the notebook as a reflection, "*I made process skill error (computational error) when using the definition (of derivates). It makes me think that providing shortcuts like the rules or formulae will be helpful, (with a note that) students have (already) understood the underlying concepts of derivatives*".

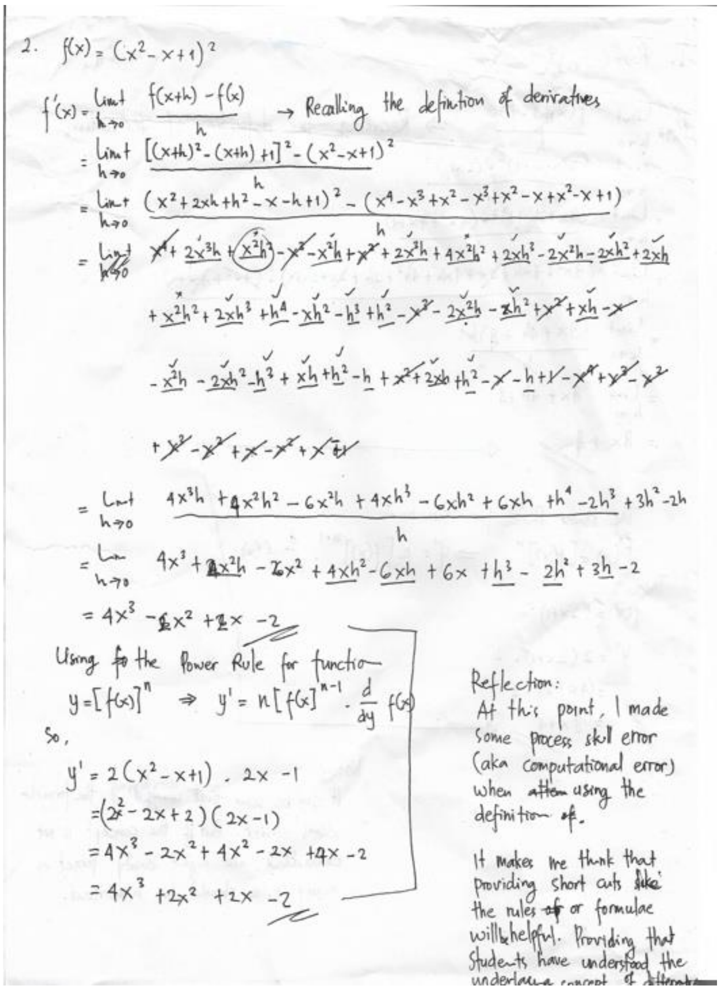

Figure 4. The advantage of procedural knowledge in solving a derivative problem

Therefore, once the underlying concepts or definition of derivatives has been understood, a rational understanding of the shortcut method can be more beneficial, which indicates the development of procedural knowledge. Hence, memorising mathematics rules or formulae which is part of procedural knowledge, is equally valuable and useful as developing conceptual knowledge. This clearly demonstrates that both procedural and conceptual are closely linked to each other (Long, 2005), but it is crucial to note that conceptual knowledge should be embedded first before offering procedural knowledge. This assertion is in line with what Hurrell (2021) explains that there is a strong likelihood of developing proper knowledge in mathematics if it starts with conceptual knowledge and then moves to procedural knowledge, not the reverse, procedural to conceptual. Having understood the underlying concepts of derivatives and being able to compute a number of mathematics problems using the shortcut method, I noticed this moment in which I gained proceptual knowledge.

Another notable reflection during my study was a shift of understanding from an informal level to a more formal level of understanding. The informal level of understanding occurred when I employed graphs in measuring the gradient of a curve as a powerful tool for me to visualise the concept of derivatives (see Figure 2). Later, I could flexibly deal with various derivatives problems without creating any graph as I have comprehended the definition of derivatives which indicates the shift to a formal level of understanding (Heuvel-Panhuizen, 2003). To summarize my overall learning journey on differentiation, I constructed the learning framework with terminologies from the relevant kinds of literature, as delineated in Figure 5.

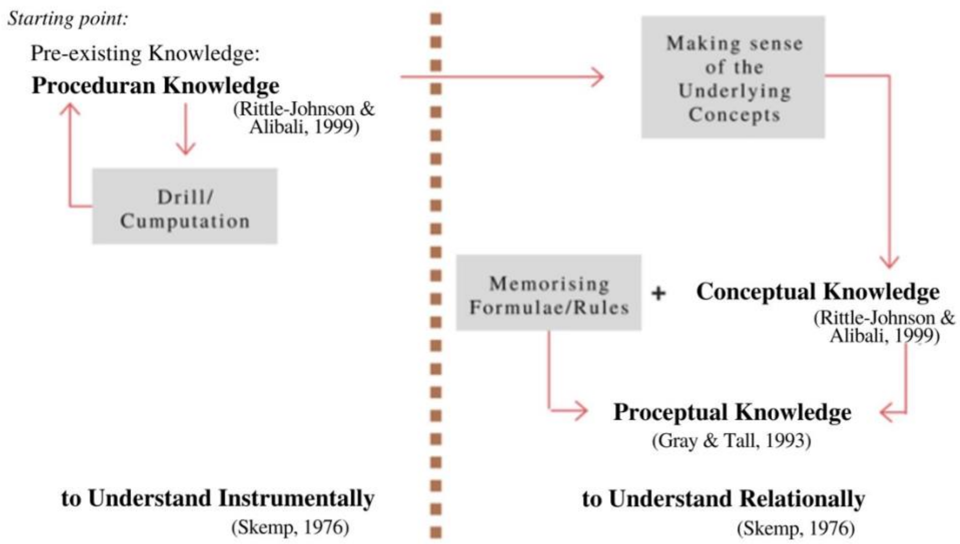

Figure 5. Learning framework on developing proceptual knowledge on derivates

The above learning framework presented in Figure 5 summarises how an ideal proceptual knowledge was developed from my independent learning of derivates. The drill and practice I did on numbers of derivatives problems did not create new knowledge but merely improved my computational skills or memorising mathematics facts and procedures. This kind of knowledge was categorised by Skemp (1976) as instrumental understanding, which to his assertion, does not indicate understanding at all. This argumentation might be accurate as challenges became evident in my learning when I encountered mathematics problems that use slightly or even completely different procedures. Therefore, I tried to comprehend the underlying concepts of the procedural knowledge of derivatives I have already memorised as well as those formulae that I have not memorised yet. By understanding the fundamental definition and the interrelationships between the concepts of derivatives, conceptual knowledge has been developed (see also Rittle-Johnson & Alibali, 1999). This, however, does not necessarily mean that conceptual knowledge is superior to procedural knowledge and ignores the role of procedural knowledge (see Hurrell, 2021; Long, 2005). In fact, procedural knowledge has a significant role in my own learning, as illustrated in Figure 4. At this point, the acquisition of both procedural and conceptual knowledge of derivates indicates the acquisition of so-called proceptual knowledge (Gray & Tall, 1992; 1994). This, in my interpretation, is similar to what Skemp (1976) termed as relational understanding, which is "knowing what to do and why" (p. 20).

After going through my independent study of differentiation, I also realised that there was a change of emotion -in a positive way- that I feel about the knowledge I developed in this module. Emotion, at this point, is simply defined as my view or perspective towards derivatives. At the beginning of my study, I honestly regretted my decision to challenge myself and take this topic which I found difficult. Even after doing some drills, I could not understand what I was doing. However, by understanding the concept relationally through the stages I went through, including proving the formulas of the derivatives, in the end, my lack of proceptual knowledge can be overcome. This also indicates the ability to prove mathematical concepts, facts, or formulae plays an important role in a way that proof establishes a logical connection in mind. This can give learners the feeling of an adequate explanation of "why" or "how" certain formula is derived, and that is intellectually satisfying (Cuoco, Goldenberg, & Mark, 1996).

The implication of the proposed framework

As the world continues to develop, there emerges an urgency to reorganise mathematics education curricula in schools to prepare learners for the future development of the world. In the mathematics context, Cuoco, Paul Goldenberg, & Mark (1996) propose that the earliest step in achieving such an intention is by providing an academic experience that allows students to develop some good habits of mind. As asserted by Skemp (1979:82) that "habits are learned, not innate. … Once established, habits are very difficult to change." This implies that to prepare students with habits, from now on, the curricula for the 21st century should be designed to create mathematical habits to make lifelong learners one of the main goals of 21st-century curricula (Demirel, 2009; Trilling & Fadel, 2009). Accordingly, in this section, some aspects or reflections from my subject study on derivatives could be a recommendation to consider in designing mathematics curricula in the 21st century.

Pull learning, not push information

One valuable reflection from my subject study on derivatives is that I, as a learner, pull learning through independent study, compared to my previous study in high school, where information was pushed to be acquired by learners. It is believed that the learning process should create a didactical environment where students engage actively in their learning endeavour to construct their knowledge or understanding of mathematics. This orientation might be linear to the theory of constructivism in which the learner is viewed as an active participant, and learning happens through the process of constructing knowledge (pulled learning) rather than merely acquiring insight or ready-made information from adults (pushed information) (Anderson et al., 2013; Duffy & Cunningham, 1996). Similarly, the 21st curriculum is designed to pull authentic learning experiences where learners are fostered to take responsibility to explore their own learning and choose the most effective strategies for their own learning (Cabi & Yalcinalp, 2012). This self-reliance learning on an ongoing basis will ultimately lead learners to create a habit or personal characteristic as lifelong learners, that is to say, having a sense of willingness or motivation to always learn. It implies that learning does not solely take place in the academic context in formal school years but rather takes place because of a sense of curiosity or awareness to feed the hungry mind regardless of time and place.

Develop relational understanding/proceptual knowledge

Although through instrumental understanding, pupils can get the answer in an easier and faster way, it virtually indicates short terms success (Skemp, 2020). Furthermore, the methods that bring about short-term success may lead to long-term failure (Gray & Tall, 1992). On the other hand, if the goal of the learners is to understand mathematics relationally, although it might take much time and frustration, it advantages them in many ways in which the knowledge gained is more adaptable to new tasks, less memorisation is involved, and it will create a feeling of confidence in their own ability (Skemp, 1976). This notion also implies that learning for 21st-century curriculum should emphasise the process of understanding rather than outcome-oriented learning. In this sense, the concept of lifelong learning does not only work to accomplish present targets but also to impart future long-term values and attitudes to learning (Demirel, 2009).

Encourage learning transfer

As stated by Cornell (1999), one of the main reasons students tend to dislike mathematics, in general, is because they think they will never employ a particular algorithm or equation outside of mathematics or school. In the 21st-century curriculum, on the contrary, students should necessarily be prepared to transfer the mathematical knowledge and skills they gain to other contexts outside of mathematics (Saavedra & Opfer, 2012). This implies that learning should provide opportunities for learners to not only make connections among mathematical ideas – as part of relational understanding – but also to recognise and apply mathematics in other disciplines or other areas of their lives. The implication of this notion might lead us to the idea of a 21st-century curriculum in which one discipline is interrelated with another (Noddings, 2007). For example, in my learning process, I tried to figure out how the concept of derivatives can be useful and applicable in solving problems in other disciplines, such as Chemistry, Physics, and Economics. In chemistry, derivatives are broadly used to predict functions like reaction rates or radioactive decay. In the field of physics, the use of concepts of derivatives is applicable to solve problems in relation to motion, electricity, heat, light, harmonics, acoustics, astronomy, and dynamics. In biology, meanwhile, to formulate rates such as birth and death rates, the concepts of derivatives are involved. In economics, the derivative is generally utilized to compute marginal cost and revenue to predict maximum profit in a specific setting. By making-connection between mathematics and other fields of science, we can expect students to realise how vital the role of mathematics is in their life as well as in their future careers after school (Evans et al., 2013).

Conclusion

Reflections from my learning journey on developing proceptual knowledge on differentiation have emerged invaluable lessons and a critical stance on how mathematics should be introduced to students. In preparing students to face future challenges, the 21st-century mathematics curriculum needs to equip students with proceptual knowledge of mathematics concepts. Pulling independent learning among students instead of pushing it is one viable way to achieve this goal. Besides, developing both procedural as well as conceptual understanding of differentiation has altered the way I study and understand mathematics concepts, and therefore, such a type of understanding needs to be nurtured among learners. Lastly, the transfer of mathematical knowledge and skills that students acquire to other contexts outside of mathematics or in a real-life situation should be encouraged as a key to preparing them for the future.

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